Primordial Perturbations Spectra in a Holographic Phase

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In this note, we assume that the universe begins in a holographic thermal equilibrium phase with the divergent correlation length, and the phase transition to the radiation phase of standard cosmology goes with the abrupt reducing of correlation length. In this case, the primordial perturbations may be induced by thermal fluctuations in this holographic phase. We calculate the spectra of this holographic primordial perturbations, and find that the scalar spectrum has a slightly red tilt and the tensor perturbation amplitude has a moderate ratio, which may be tested in coming observations. The results plotted in $r - n_s$ plane is similar to that of large field inflation models. However, for fixed efolding number, they are generally in different positions.

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Recently, it has been shown in Ref. [1] that the string thermodynamic fluctuation may lead to a scale invariant spectrum of scalar metric perturbation, see Ref. [2, 3, 4] for more details and also Ref. [5] for criticisms. The key point obtaining the scale invariance lies in the specific heat for a fixed region be proportional to the area, which is a significant feature of string thermodynamics [6].

In Ref. [7], it was noted that the specific heat scaling as the area may also be induced in a high temperature phase depicted in terms of holographic principle. The holographic principle [8, 9], which states that the fundamental degrees of freedom of a physical system are bound by its surface area, is generally taken as a fundamental property of the microscopic theory of quantum gravity. In thermal equilibrium phase with the temperature $\mathcal{T}$, which obeying the holographic principle, to obtain the metric perturbations in various wavelength, one need to calculate the fluctuations of the energy momentum tensor on various length R, up to the maximal correlation length. For each value of R, the holographic phase means that the space region bounded by a surface $\sim R^2$ can be described by a finite number of degrees of freedom given by $\pi R^2/G$. These fundamental degree of freedom will be expected to have the energy $\mathcal{T}$. In this case the total energy may be written as $\mathcal{E} = \pi R^2 \mathcal{T}/G$, which gives the special heat $c_R = \partial\mathcal{E}/\partial\mathcal{T} \sim R^2$. Thus dependent of the existence of an early holographic phase, one obtain a specific heat scaling as the area, which straightly leads to the scale invariance of scalar spectrum [7].

The primordial perturbation arising from a thermal holographic phase provides an alternative to the seeding mechanisms responsible for the structure formation of observable universe. Thus it is interesting and also significant to ask what is the general features of holographic primordial perturbations, which is testable in coming observations? This will be done in this note. To make the discussions here be model independent as possible, we will be not constrained to some concrete model, especially the details of phase transition. We only assume that the holographic phase has a nearly divergent correlation length, which is required to assure all interesting modes observed today are in causal contact before phase transition. The phase transition to the radiation phase of standard cosmology goes with the abrupt reducing of correlation length. In this case, the primordial perturbation may be induced by thermal fluctuations in this holographic phase.

The phase transition of radiation phase to holographic phase is characterized by the divergence of the correlation length of metric perturbation. The correlation length is generally given by $l \sim c_s h^{-1}$, where c_s denotes the sound speed of metric perturbation, and h is the comoving Hubble parameter. Though how to introduce the rapid change of correlation length of metric perturbation in a holographic phase is open, it seems that the divergence of correlation length suggests either c_s diverges or $h \simeq 0$. The case of $h \simeq 0$ may occur e.g. during bounce, e.g. see Ref. [10] for a relevant review and also Refs. [11, 12] for recent examples. However, the phase transition relevant to holographic phase dose not need to involve the bounce of background. Thus we will consider the case of divergent c_s. Though the divergence of c_s can hardly be understood in Einstein gravity and is generally expected to have a profound origin, for a phenomenological discussion we may implemented it in an effective theory, see Ref. [13], in which the perturbation equation is modified with a varying sound speed, while the evolution of background equation is not changed and is still that of Einstein gravity. This means when $\mathcal{T} \sim \mathcal{T}_c$, where $\mathcal{T}_c$ is the critical temperature of phase transition, c_s rapidly approaches infinity, while $\mathcal{T} \ll \mathcal{T}_c$, it is one.

When c_s is very large, we can have $c_s k \gg h$, which means that the perturbations are deep in the sound horizon. Thus in the longitudinal gauge, see Ref. [14] for details on the theory of cosmological perturbations, the (00) equation of metric perturbation may be reduced to

$$c_s^2 k^2 \Phi \simeq a^2 G \delta\rho, \tag{1}$$

where Φ denotes the scalar metric perturbation and G is the Newton constant, and the terms relevant to h has been neglected since $c_s k \gg h$. Eq.(1) is Poisson like, but is in relativistic sense. From Eq.(1), we have the scalar perturbation spectrum

$$\mathcal{P}_\Phi(k) \simeq \frac{a^4 G^2}{c_s^4 k^4} \mathcal{P}_{\delta\rho}(k) = \frac{a^4 G^2}{c_s^4 k^4} < \delta\rho^2 >, \tag{2}$$

where $<\delta\rho^2>$ is the mean square fluctuation of the energy density, which need to be calculated in each fixed length scale R, up to the physical Hubble scale a/h, since what we introduce is only the nearly divergence of correlation length of metric perturbation, while that of matter fluctuation is still limited by Hubble scale. Though here the physical wavelength a/k of matter fluctuations required to induce the metric perturbation at present observable scale are sub sound horizon scale, they are generally super Hubble scale, i.e. $k < h$, see the red solid lines in Fig.1. Thus in principle we are not able to deduce the metric perturbation by calculating the matter fluctuation, since in super Hubble scale the matter fluctuations has frozen. However, if the sound speed during phase transition is nearly diverged, the case will be different. In this case the effective physical wavelength of metric perturbation will obtain a strong suppress $\sim 1/c_s$ and become $a/(c_sk)$, which may be sub Hubble scale well, see the red dashed line in Fig.1. Thus it seems reasonable to replace R with $a/(c_sk)$ in $<\delta\rho^2>$, see Ref. [13] for a more detailed analysis. In thermal holographic phase,

$$<\delta\rho^2>(R=\frac{a}{c_sk}) = -\frac{1}{R^6}\frac{\partial}{\partial\beta}\left(\mathcal{F}+\beta\frac{\partial\mathcal{F}}{\partial\beta}\right) = \frac{\pi\mathcal{T}^2}{GR^4}. \quad (3)$$

where $\beta = 1/\mathcal{T}$, and $\mathcal{F}$ is the free energy, which is given as

$$\mathcal{F} = \frac{\pi R^2\mathcal{T}}{G}\ln\left(\frac{\mathcal{T}_c}{\mathcal{T}}\right), \quad (4)$$

by the energy $\mathcal{E} \equiv \mathcal{F}+\beta\left(\frac{\partial\mathcal{F}}{\partial\beta}\right)$. We substitute Eq.(3) into Eq.(2), then can obtain

$$\mathcal{P}_\Phi(k) \simeq G\mathcal{T}^2. \quad (5)$$

From Eq.(5), one can see that the spectrum of scalar metric perturbation is scale invariant.

When c_s is decreasing, the metric perturbation with some scale k^{-1} can leave the sound horizon, and quickly freeze. The spectrum of curvature perturbation ξ in comoving supersurface $\mathcal{P}_\xi \simeq \mathcal{P}_\Phi$ up to a factor with order one, which is constant in the super horizon scale. Thus the spectrum of the comoving curvature perturbation can be nearly scale invariant and its amplitude can be calculated at the time when the perturbation exits the sound horizon, i.e. $k = h/c_s$, which gives the value of $\mathcal{T}$ in Eq.(5) at the sound horizon crossing.

The sound horizon crossing requires $k = h/c_s$, thus we can obtain

$$\frac{k_e}{k} \simeq c_s \equiv \left(\frac{\mathcal{T}_c}{\mathcal{T}_c-\mathcal{T}}\right)^p, \quad (6)$$

where the subscript 'e' denotes the end time of phase transition and thus $c_{s(e)} \simeq 1$, and we also neglected the change of h during phase transition. When we include

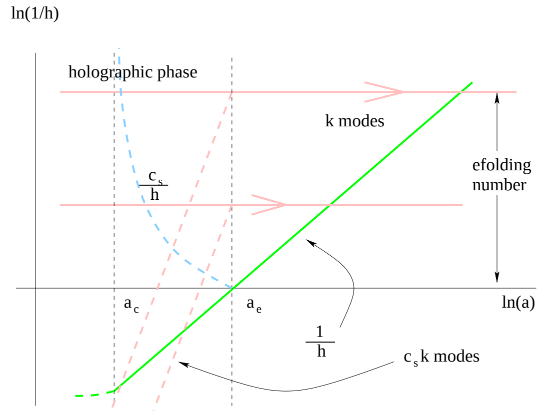

FIG. 1: The sketch of evolution of the correlation length or sound horizon $\ln(c_s/h)$ and the Hubble horizon $\ln(1/h)$ with respect to the scale factor $\ln a$. During the transition, the perturbations are able to leave the sound horizon and become nearly scale invariant primordial perturbations responsible for the structure formation of observable universe.

the change of h, there will be a factor like $(\mathcal{T}/\mathcal{T}_e)^n$ before the right hand term of Eq.(6), where $n \sim \mathcal{O}(1)$, which is negligible when being compared to that of c_s. In Eq.(6), c_s is taken by using the analysis in condensed matter physics, where $p > 0$ denotes the critical exponent of transition. By using Eq.(6), the spectrum (5) of metric perturbation can be rewritten as

$$\mathcal{P}_\Phi \simeq G\mathcal{T}_c^2\left[1-\left(\frac{k}{k_e}\right)^{1/p}\right]^2. \quad (7)$$

Note that k_e is the last mode to be generated, thus we have $k < k_e$, especially on large scale $k \ll k_e$. Thus the amplitude is approximately $\mathcal{P}_\Phi \simeq G\mathcal{T}_c^2$. The spectral index is given by

$$n_s - 1 = \frac{-2\left(\frac{k}{k_e}\right)^{1/p}}{p\left[1-\left(\frac{k}{k_e}\right)^{1/p}\right]}. \quad (8)$$

We can see that the scalar spectrum is red and its tilt is determined by the critical exponent and the temperature at horizon crossing, since k is related to $\mathcal{T}$ by Eq.(6).

We define

$$\mathcal{N} \equiv \ln\left(\frac{k_e}{k}\right), \quad (9)$$

which measures the efolding number of mode with some scale $\sim k^{-1}$ which leaves the horizon before the end of phase transition. When taking the comoving Hubble parameter $h = h_0$, where the subscript '0' denotes the present time, we generally have $\mathcal{N} \sim 50$, which is required by observable cosmology. From Eq.(6), we can see that when $\mathcal{T} \to \mathcal{T}_c$, k_e/k nearly approaches to infinity, thus the efolding number is actually always enough

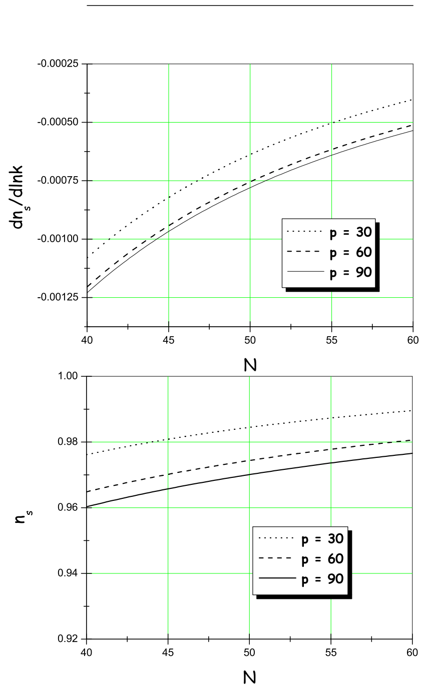

FIG. 2: The figure of the scalar spectral index n_s and its running $dn_s/d\ln k$ with respect to the efolding number $\mathcal{N}$ for different critical exponent p.

as long as the initial volume of holographic phase is large enough. We substitute Eq.(9) into Eq.(8) and can obtain the scalar spectral index

$$n_s - 1 = -\frac{2}{p}\cdot\left(\frac{1}{e^{\mathcal{N}/p}-1}\right). \tag{10}$$

The running of spectral index is given by

$$\frac{dn_s}{d\ln k} = \frac{n_s-1}{p}\cdot\left(\frac{1}{1-e^{-\mathcal{N}/p}}\right), \tag{11}$$

which is the same as that in Ref. [7] only when the efolding number $\mathcal{N} \gg p$. The figure of n_s and $dn_s/d\ln k$ with respect to the efolding number $\mathcal{N}$ for different critical exponent p are plotted in Fig.2. It seems that to fit the observations [16], p should be very large. Substituting Eq.(10) into Eq.(11), we can obtain $dn_s/d\ln k \sim 1/p^2$, which means that the running is quite small, which may also be seen in Fig.2.

Then we will calculate the tensor perturbation spectrum. Though it is not necessary that the divergence of correlation length of tensor perturbation is same with that of scalar metric perturbation, here we assume that they share same one, like Eq.(6). In principle the tensor perturbation may be produced from the vacuum without any source, as happens during inflation. However, when the source is included, its generation mechanism will be quite different. In this case the source term will master the equation of tensor perturbation. When $c_s k \gg h$, similar to the analysis of scalar metric perturbation, we can find that in left hand side of the equation of tensor perturbation, compared with $c_s^2 k^2 h_{ij}(k)$, other relevant terms with h_{ij} may be neglected. For example, in unit of Hubble time h''_{ij} may be written as $h^2\Delta h_{ij}$, thus we have $h''_{ij} \simeq h^2\Delta h_{ij} \ll c_s^2 k^2 h_{ij}$. Thus the equation of tensor perturbation may be reduced to

$$c_s^2 k^2 h_{ij}(k) \simeq a^2 G\delta\mathcal{T}_{ij}(k), \tag{12}$$

where h_{ij} is the tensor perturbation and can be expanded in term of the two basic traceless and symmetric polarization tensors $e^{\pm}_{ij}$ as $h_{ij} = \sum h_{\pm}e^{\pm}_{ij}$. Thus we have the tensor perturbation spectrum

$$\mathcal{P}_{\mathrm{T}} \simeq \frac{a^4G^2}{c_s^4k^4}C^{i\ i}_{\ j\ j}(R = \frac{a}{c_s k}). \tag{13}$$

where $C^{i\ i}_{\ j\ j}$ is the offdiagonal spatial parts of the correlation function of energy momentum tensor. However, their order of magnitude may be estimated by the order of magnitude of diagonal parts

$$C^{i\ i}_{\ i\ i} = \frac{1}{R^2\beta}\frac{\partial p}{\partial R} = \frac{\pi\mathcal{T}^2}{GR^4}\ln\left(\frac{\mathcal{T}_c}{\mathcal{T}}\right), \tag{14}$$

which is a good approximation, as was pointed out in Ref. [15], where the pressure p can be obtained by the free energy $p \equiv -\frac{\partial\mathcal{F}}{R^3\partial(\ln R)}$ with Eq.(4), see Refs. [3, 15] for details. Thus we may have

$$\mathcal{P}_{\mathrm{T}} \simeq G\mathcal{T}^2\ln\left(\frac{\mathcal{T}_c}{\mathcal{T}}\right). \tag{15}$$

Note that when $\mathcal{T} = \mathcal{T}_c$, $\ln(\mathcal{T}_c/\mathcal{T}) = 0$. However, this does not means that for present observable scale, $\mathcal{P}_{\mathrm{T}} = 0$ since for fixed efolding number, generally $\mathcal{T} < \mathcal{T}_c$, see Fig.1. Thus dependent on the degree of $\mathcal{T} \to \mathcal{T}_c$, the tensor perturbation will have a moderately decreasing amplitude, which means that its spectrum is slightly blue tilt. This can also be seen as follows

$$\begin{aligned}\mathcal{P}_{\mathrm{T}} &\simeq G\mathcal{T}_c^2\left[1-\left(\frac{k}{k_e}\right)^{1/p}\right]^2\ln\left[1-\left(\frac{k}{k_e}\right)^{1/p}\right]^{-1} \\ &\simeq G\mathcal{T}_c^2\left(\frac{k}{k_e}\right)^{1/p}, \end{aligned} \tag{16}$$

where Eq.(6) and $k \ll k_e$ on large scale have been used in the first and second lines, respectively. Thus the spectral index of tensor spectrum $n_{\mathrm{T}} \simeq 1/p$, which is slightly blue. This is similar to the case in string gas mechanism [15], but different from that of slow-roll inflation in which the tilt of tensor spectrum is generally red.

The ratio r of tensor to scalar is an important quantity for observation, which as well as n_s makes up of the $r - n_s$ plane [17, 18], in which different classes of inflation modes are placed in different regions. For the holographic primordial perturbations, we have the ratio of tensor to scalar

$$r \equiv \frac{\mathcal{P}_{\rm T}}{\mathcal{P}_\xi} = C \ln\left(\frac{\mathcal{T}_c}{\mathcal{T}}\right), \tag{17}$$

where Eqs.(5) and (15) have been used, and $C \sim \mathcal{O}(1)$ is expected to be a constant. Note that Eqs.(1) and (12) are valid only in the approximation $c_s k \gg h$, thus in principle we need to solve full equation of Φ and h_{ij} to determined the value of C. Besides, C is also determined by some exact relations between the correlation lengths of scalar mode and tensor mode, between the metric perturbation $\mathcal{P}_\Phi$ and the curvature perturbation $\mathcal{P}_\xi$, between the magnitudes of diagonal spatial parts $C^{i\ i}_{\ i\ i}$ and off-diagonal spatial parts $C^{i\ i}_{\ j\ j}$ of the correlation function of energy momentum tensor. We substitute Eqs.(6) and (17) into Eq.(8) and can obtain

$$n_s - 1 = -\frac{2}{p}\cdot\left(e^{r/C} - 1\right). \tag{18}$$

Thus for a definite C, the relation between n_s and r is only dependent of the critical exponent p. In principle, for different C, one always may maintain the qualitative behavior of $r - n_s$ lines by changing the value of p. To match it to observations, we can note that $C > 1$ is obviously not favored since this will lead to a quite large tensor perturbation. Here no loosing generality, we set $C = 0.2$. The $r - n_s$ figure for different critical exponent is plotted in Fig.3. The black square dots in Fig.3 denote those with the efolding number $\mathcal{N} = 50$, which can fit recent observations [16] well, see also Refs. [19] and recent [20] for discussions on the bounds on $r - n_s$.

The lines in Fig.3 lies in the region of large field inflation models, see e.g. Ref. [21] for the discussions on various inflation models and also recent Ref. [22], which suggests that the holographic primordial perturbations may has similar prediction with large field inflation models. However, they can be distinguished by the efolding number. For large field inflation models $\sim \phi^p$, in the slow-roll approximation, for fixed efolding number, the larger p is, the more the spectral index n_s deviates from scale invariant. The case with least deviation corresponds to $p = 2$, which is $m^2\phi^2$ model, in which $n_s - 1 \simeq -(2/\mathcal{N})$. While in the holographic primordial perturbations discussed here, with Eq.(10), for small p, there will a suppression from $e^{\mathcal{N}/p}$, thus the maximal deviation from scale invariant occurs only when $\mathcal{N} \ll p$. In this case, we may expand the exponent in the numerator of Eq.(10) to the second order $\mathcal{N}/p$ and obtain

$$n_s - 1 \simeq -\frac{2}{\mathcal{N}}\cdot(1 - \frac{\mathcal{N}}{2p}). \tag{19}$$

We can see that the deviation is always small than $-(2/\mathcal{N})$. From Eq.(18), $\mathcal{N} \ll p$ means that to obtain an enough deviation from scale invariant the ratio r of tensor to scalar should be very large, which seems be not favored by observations.

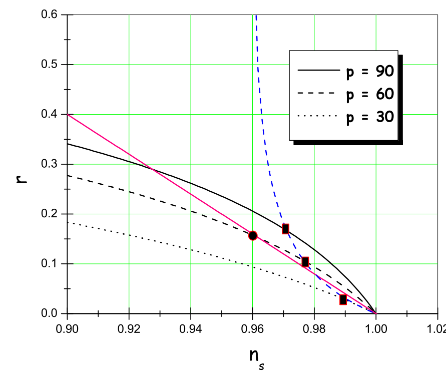

FIG. 3: The $r - n_s$ figure with $C = 0.2$. The blue dashed line denotes the line with the efolding number $\mathcal{N} = 50$, which crosses the lines with $p = 30, 60, 90$ with the crossing dots denoted by the black square. The red line correspond to that of $m^2\phi^2$ inflation model, in which the black round dot denotes that with $\mathcal{N} = 50$

Therefore, generally in the region of large field inflation models of $r - n_s$ plane, for fixed efolding number, the deviation of holographic primordial perturbation spectra from scale invariant is always smaller than that of large field inflation models, which is independent of the value of C. This can be also seen in blue dashed line in Fig.3, which is plotted by combining Eqs.(10) and (18) in which p can cancelled. This line reflects the relation between r and n_s for fixed efolding number $\mathcal{N}$, in which $\mathcal{N} = 50$ has been taken. This result provides an interesting smoking gun, which means that for $\mathcal{N} > 50$, see Ref. [23] for a review on the value of $\mathcal{N}$, if the observations give $n_s < 0.96$, then the holographic primordial perturbations will be ruled out.

In summary, we calculate the primordial perturbation spectra arising from a holographic phase with the assumption that the correlation length of metric perturbation diverges at the critical point of phase transition. The results shown in $r - n_s$ plane are in the range of large field inflation models with a slightly red tilt of scalar spectrum and a moderate ratio of tensor to scalar amplitude. However, for fixed efolding number, they are generally in different positions. The qualitative features of perturbation spectra are only dependent of the thermodynamics of holographic phase and the change of correlation length during phase transition, not the details of phase transition. In this sense the predictions of holographic primordial perturbation given in this note are unambiguous, which may be tested in coming observations.

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